Postulates of quantum mechanics, as stated by various authors

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About this document

Quantum mechanics is the most fundamental theory of physics. As such, there are no known principles or truths of physics that logically precede those lying at the foundational level of QM itself. The foundational propositions of QM must first be accepted as postulates, if the body of knowledge that is physics, taken in its entirety, is to remain logically consistent and physically meaningful.

Owing to their roots in inductive reasoning performed within a certain phenomenological context, and not in deductive reasoning as within the relatively narrower confines of mathematics, the postulates of QM cannot be regarded as mathematical axioms—whether in terms of form or function.

The postulates should instead be seen as a minimalist set of propositions that serve as the conceptual anchor-points, or core nodes in concept-map, in reference to which our knowledge of the physical world may best be organized. The sheer scope and variety of physical phenomena being encapsulated makes it impossible to impart the mathematical kind of rigour to the postulates of QM. In other words, there cannot be a Euclid for QM.

As to the current state of this topic, viz., Foundations of QM, while there indeed is
a very good general agreement on what facts of nature must be encapsulated within these postulates if they are to serve their foundational purpose, the specific formulations put forth by various authors do differ significantly in terms of their detail. Almost a century after the currently accepted theory of QM was completed, at least in terms of the mathematical principles lying at its core, the process of refining the statements for its postulatory basis still goes on—at times, with good reasons too!

The variety of available formulations can easily get confusing to the beginning student. At least, such was my own experience. That’s why, during my studies and research, I decided to compile in one place a few of these differing sets of statements. The result is this document.

Almost all of this document consists of verbatim reproduction from the original sources. My own contribution is limited to one or two editorial remarks; these have been explicitly noted as such.

This is very much a work in progress. Typographical errors are easily possible. Therefore, it is advisable not to rely on this document fully, but instead to make reference to the original sources themselves. Needless to add, copyrights and any other rights belong to the respective owners of the original sources.

If you spot some errors that crept in during the compilation process, then please do not hesitate to let me know. Thanks in advance.

*NB:* This document may change at any time without notice. (Indeed, in future, I do plan to include the formulations by several more authors too.) Hence, for any correspondence or reference purposes, the time-stamp of the *\LaTeX* compilation should be taken as the “edition” or “version” number.

—ARJ
Pune, India
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1. Associated with any particle moving in a conservative field of force is a wave function which determines everything that can be known about the system.

2. With every physical observable $q$ there is associated an operator $Q$, which when operating upon the wavefunction associated with a definite value of that observable will yield that value times the wavefunction.

3. Any operator $Q$ associated with a physically measurable property $q$ will be Hermitian.

4. The set of eigenfunctions of operator $Q$ will form a complete set of linearly independent functions.

5. For a system described by a given wavefunction, the expectation value of any property $q$ can be found by performing the expectation value integral with respect to that wavefunction.

6. The time evolution of the wavefunction is given by the time dependent Schrodinger equation.
2  McQuarrie[2]

1. **Postulate 1: The state of a system is completely specified by its wavefunction.** The state of a quantum-mechanical system is completely specified by a function $\Psi(\vec{r}, t)$ that depends on the coordinates of the particle and on the time. Thus function, called the wave function or the state function, has the important property that $\Psi^*(\vec{r}, t) \Psi(\vec{r}, t) \, dx \, dy \, dz$ is the probability that the particle lies in the volume element $dx \, dy \, dz$, located at $\vec{r}$, at the time $t$.

2. **Postulate 2: Quantum-mechanical operators represent classical-mechanical variables:** To every observable in classical mechanics there corresponds a linear operator in quantum mechanics.

**Postulate 2’: Quantum-mechanical operators must be Hermitian operators:** To every observable in classical mechanics there corresponds a linear Hermitian operator in quantum mechanics.

3. **Postulate 3: Observable quantities must be eigenvalues of quantum-mechanical operators:** In any measurement of the observable associated with the operator $\hat{A}$, the only values that will ever be observed are the eigenvalues $a$, which satisfy the eigenvalue equation

$$\hat{A}\Psi = a\Psi$$

4. **Postulate 4: The average value of an observable quantity:** If a system is in a state described by a normalized wave function $\Psi$, then the average value of the observable corresponding to $\hat{A}$ is given by

$$\langle a \rangle = \int_{-\infty}^{\infty} \Psi^* \hat{A}\Psi \, d\tau$$

5. **Postulate 5: The time-dependence of wave functions is governed by the time dependent Schrödinger equation:** The wave function or state function of a system evolves in time according to the time-dependent Schrödinger equation

$$\hat{H}\Psi(x, t) = i\hbar \frac{\partial\Psi}{\partial t}$$

6. **Additional postulate by C. David Sherrill[3]:** The total wavefunction must be antisymmetric with respect to the interchange of all coordinates of one fermion with those of another. Electronic spin must be included in this set of coordinates.
3 Atkins and Friedman[4]

1. **Postulate 1: States and wavefunction:** The state of a system is fully described by a function \( \Psi(r_1, r_2, \ldots, t) \).

2. **Postulate 2: The fundamental prescription:** Observables are represented by Hermitian operators chosen to satisfy the commutation relations

\[
[q, p_{q'}] = i \hbar \delta_{qq'} \quad [q, q'] = 0 \quad [p_{q}, p_{q'}] = 0
\]

where \( q \) and \( q' \) each denote one of the coordinates \( x, y, z \) and \( p_{q} \) and \( p_{q'} \) the corresponding linear momenta.

3. **Postulate 3: The outcome of measurements:** When a system is described by a wavefunction \( \psi \), the mean value of the observable \( \Omega \) in a series of measurements is equal to the expectation value of the corresponding operator.

\[
\langle \Omega \rangle = \frac{\int \psi^* \hat{\Omega} \psi \, d\tau}{\int \psi^* \psi \, d\tau} = \frac{\langle \psi | \Omega | \psi \rangle}{\langle \psi | \psi \rangle}
\]

**Postulate 3’:** When \( \psi \) is an eigenfunction of the operator \( \Omega \), the determination of the property \( \Omega \) always yields one result, namely the corresponding eigen-value \( \omega \). The expectation value will simply be the eigenvalue \( \omega \). When \( \psi \) is not an eigenfunction of \( \Omega \), a single measurement of the property yields a single outcome which is one of the eigenvalues of \( \Omega \), and the probability that a particular eigenvalue \( \omega_n \) is measured is equal to \( |c_n|^2 \), where \( c_n \) is the coefficient of the eigenfunction \( \psi_n \) in the expansion of the wavefunction. Moreover, immediately after that measurement, the state of the system will be \( \psi_n \).

4. **Postulate 4: The interpretation of the wavefunction (the Born interpretation):** The probability that a particle will be found in the volume element \( d\tau \) at the point \( r \) is proportional to \( |\psi(r)|^2 \).

5. **Postulate 5: The equation for the wavefunction:** The wavefunction \( \psi(r_1, r_2, \ldots, t) \) evolves in time according to the equation

\[
i \hbar \frac{\partial \psi}{\partial t} = H \psi
\]
1. **Postulate 1: The wavefunction:** For every dynamical system there exists a wavefunction that is a continuous, square-integrable, single-valued function of the parameters of the system and of time, and from which all possible predictions about the physical properties of the system can be obtained. (The parameters may be the coordinates of all the particles of a many-body system and may include internal variables such as spin.)

2. **Postulate 2: The dynamical variables:** Every dynamical variable may be represented by a Hermitian operator whose eigenvalues represent the possible results of carrying out a measurement of the value of the dynamical variable. Immediately after such a measurement, the wavefunction of the system will be identical to the eigenfunction corresponding to the eigenvalue obtained as a result of the measurement.

3. **Postulate 3: The dynamical variables (position and momentum):** The operators representing the position and momentum of a particle are $\vec{r}$ and $-i\hbar \nabla$ respectively. Operators representing other dynamical quantities bear the same functional relation to these as do the corresponding classical quantities to the classical position and momentum variables.

4. **Postulate 4: Probability distributions:** When a measurement of a dynamic variable represented by the Hermitian operator $\hat{Q}$ is carried out on a system whose wavefunction is $\psi$, then the probability of the result being equal to a particular eigenvalue $q_m$ will be $|a_m|^2$, where $\psi = \sum_n a_n \phi_n$ and the $\phi_n$ are the eigenfunctions corresponding to the eigenvalues $q_n$.

5. **Postulate 5: The time dependence of the wavefunction:** Between measurements, the development of the wavefunction with time is governed by the time-dependent Schrödinger equation.

**Remarks by ARJ** In this author’s notation (p. 62), $\Psi$ is a general solution to the TDSE; $u$ (sometimes with a subscript) is the time-independent part of the wavefunction of a system in a state of given energy; $\psi$ is a general wavefunction whose time dependence is not being explicitly considered; and $\phi$ (often with a subscript) is used when the system is in an eigenstate—i.e. when some dynamical quantity (not necessarily the energy) has a known value.
5 Dorabantu[6]

1. To every state of a physical system there is a function $\Psi$ ascribed to and defining
the state.

2. If $H_1$ is the Hilbert space associated with the physical system $S_1$, and $H_2$ is the
Hilbert space corresponding to the other physical system $S_2$, then the composite
system $S_1 + S_2$ will be associated with the tensor product of the two Hilbert
vector spaces $H_1 \otimes H_2$.

3. To every observable of a physical system is associated a self-adjoint (or Hermitian)
operator allowing a complete set of eigenfunctions.

4. The time evolution of a quantum state is governed by a unitary transformation.
If $\Psi(t)$ is the probability amplitude of a quantum state at time $t$, then $\Psi(t + \Delta t)$
is its probability amplitude at a later time $t + \Delta t$, so that $|\Psi(t + \Delta t)\rangle = U(t + \Delta t)|\Psi(t)\rangle$, where $U$ is a unitary linear operator.

5. As a result of a measuring process performed upon an observable $F$, we will ob-
tain only the eigenvalues of the Hermitian operator, $F$, associated to the observ-
able. The probability of getting an eigenvalue $f_k$ corresponding to the discrete
spectrum is $c_k$, and the probability of getting an eigenvalue $f_\alpha$ corresponding to
the continuous spectrum within an interval $d\alpha$ is $|c_\alpha|^2 d\alpha$. 
6 Shankar[7]

1. Postulate I

- **Classical mechanics:** The state of a particle at any given time is specified by the two variables \( x(t) \) and \( p(t) \), i.e., as a point in a two-dimensional phase space.

- **Quantum mechanics:** The state of the particle is represented by a vector \( \psi(t) \) in a Hilbert space.

2. Postulate II

- **Classical mechanics:** Every dynamical variable \( \omega \) is a function of \( x \) and \( p \):
  \[
  \omega = \omega(x, p).
  \]

- **Quantum mechanics:** The independent variables \( x \) and \( p \) of classical mechanics are represented by Hermitian operators \( X \) and \( P \) with the following matrix elements in the eigenbasis of \( X \). (\( X \) and \( P \) are discussed in section 1.10 of the book.)
  \[
  \langle x | X | x' \rangle = x \delta(x - x')
  \]
  \[
  \langle x | P | x' \rangle = -i\hbar \delta'(x - x')
  \]

  The operators corresponding to the dependent variables \( \omega(x, p) \) are given by Hermitian operators
  \[
  \Omega(X, P) = \omega(x \to X, p \to P)
  \]
  (by which we mean that \( \Omega \) is the same function of \( X \) and \( P \) as \( \omega \) is of \( x \) and \( p \).)

3. Postulate III

- **Classical mechanics:** If the particle is in a state given by \( x \) and \( p \), the measurement of the variable \( \omega \) will yield a value \( \omega(x, p) \). The state will remain unaffected.

- **Quantum mechanics:** If the particle is in a state given by \( x \) and \( p \), the measurement of the variable (corresponding to) \( \Omega \) will yield one of the eigenvalues \( \omega \) with probability
  \[
  P(\omega) \propto |\langle \omega | \psi \rangle|^2.
  \]
  The state of the system will change from \( |\psi\rangle \) to \( |\omega\rangle \) as a result of the measurement. Here the measurement means, as in an ideal experiment consistent with the theory. It is assumed you are familiar with the idea of classical measurement which
can determine the state of the system without disturbing it in any way. A discussion of ideal quantum measurement follows.

4. Postulate IV

• Classical mechanics: The state variables change with time according to Hamilton’s equations:

\[
\dot{x} = \frac{\partial H}{\partial p} \\
\dot{p} = -\frac{\partial H}{\partial x}
\]

• Quantum mechanics: The state vector \(|\psi(t)\rangle\) obeys the Schrödinger equation

\[
\frac{i\hbar}{\hbar} \frac{d}{dt} |\psi(t)\rangle = H|\psi(t)\rangle
\]

where \(H(X, P) = \mathcal{H}(x \to X, p \to P)\) is the quantum Hamiltonian operator and \(\mathcal{H}\) is the Hamiltonian for the corresponding classical problem.

Remarks by ARJ The original text itself has the time-derivative in (1) as ordinary and not partial.
7 Nottale and Célérier[8]

7.1 Main postulates

1. **Postulate 1: Complex state function.** Each physical system is described by a state function which determines all can be known about the system. The coordinate realization of this state function, the wavefunction \( \psi(r, s, t) \), is an equivalence class of complex functions of all the classical degrees of freedom generically noted \( r \), of the time \( t \) and of any additional degrees of freedom such as spin \( s \) which are considered to be intrinsically quantum mechanical. Two wavefunctions represent the same state if they differ only by a phase factor (this part of the ‘postulate’ can be derived from the Born postulate, since, in this interpretation, probabilities are defined by the squared norm of the complex wavefunction and therefore the two wavefunctions differing only by a phase factor represent the same state). The wavefunction has to be finite and single valued throughout position space, and furthermore, it must also be a continuous and continuously differentiable function [with some remarks].

2. **Postulate 2: Schrödinger equation.** The time evolution of the wavefunction of a non-relativistic physical system is given by the time-dependent Schrödinger equation

\[
\frac{i \hbar}{\partial t} \frac{\partial \psi}{\partial t} = \hat{H}\psi
\]

where the Hamiltonian \( \hat{H} \) is a linear Hermitian operator, whose expression is constructed from the correspondence principle.

3. **Postulate 3: Correspondence principle.** To every dynamical variable of classical mechanics there corresponds in quantum mechanics a linear, Hermitian operator, which, when operating upon the wavefunction associated with a definite value of that observable (the eigenstate associated to a definite eigenvalue), yields this value times the wavefunction. The more common operators occurring in quantum mechanics for a single particle are listed below and are constructed using the position and momentum operators.
Position \( r(x, y, z) \)  
Multiply by \( r(x, y, z) \)

Momentum \( p(p_x, p_y, p_z) \)  
\(-i\hbar \nabla\)

Kinetic energy \( T = \frac{p^2}{2m} \)  
\(-\frac{\hbar^2}{2m} \Delta\)

Potential energy \( \Phi(r) \)  
Multiply by \( \Phi(r) \)

Total energy \( E = T + \Phi \)  
\( i\hbar \frac{\partial}{\partial t} = \Phi(r) - \frac{\hbar^2}{2m} \Delta\)

Angular momentum \( (l_x, l_y, l_z) \)  
\(-i\hbar r \times \nabla\)

More generally, the operator associated with the observable \( A \) which describes a classically defined physical variable is obtained by replacing in the ‘properly symmetrized’ expression of this variable the above operators for \( r \) and \( p \). This symmetrization rule is added to ensure that the operators are Hermitian and therefore that the measurement results are real numbers.

However, the symmetrization (or Hermitization) recipe is not unique. As an example, the quantum-mechanical analogue of the classical product \((p_x)^2\) can be either \((p_x^2 + x^2 p_x^2)/2\) or \([xp + px]/2\)^2. The different choices yield corrections of the order of some \( \hbar \) power and, in the end, it is the experiments that decide which is the correct operator. This is clearly one of the main weaknesses of the axiomatic foundation of quantum mechanics, since the ambiguity begins with second orders, and therefore concerns the construction of the Hamiltonian itself.

4. **Postulate 4: Von Neumann’s postulate.** If a measurement of the observable \( A \) yields some value \( a_i \), the wavefunction of the system just after the measurement is the corresponding eigenstate \( \psi_i \) (in the case that \( a_i \) is degenerate, the wavefunction is the projection of \( \psi \) onto the degenerate subspace).

5. **Postulate 5: Born’s postulate: probabilistic interpretation of the wavefunction.** The squared norm of the wavefunction \(|\psi|^2\) is interpreted as the probability of the system of having values \((r, s)\) at time \( t \). This interpretation requires that the sum of the contributions \(|\psi|^2\) for all values of \((r, s)\) at time \( t \) be finite, i.e., the physically acceptable wavefunctions are square integrable. More specifically, if \( \psi(r, s, t) \) is the wavefunction of a single particle, \( \psi^*(r, s, t)\psi(r, s, t) \, dr \) is the probability that the particle lies in the volume element \( dr \) located at \( r \) at time \( t \). Because of this interpretation and since the probability of finding a single particle somewhere is \( 1 \), the wavefunction of this particle must fulfil the normalization condition

\[
\int_{-\infty}^{\infty} \psi^*(r, s, t)\psi(r, s, t) \, dr = 1.
\]
7.2 Secondary postulates

One can find in the literature other statements which are often presented as ‘postulates’ but which are mere consequences of the above five ‘main’ postulates. We examine below some of them and show how we can derive them from these ‘main’ postulates.

1. **Superposition principle.** It states that a linear combination of state functions of a given physical system is a state function of this system.

2. **Eigenvalues and eigenfunctions.** Any measurement of an observable \( A \) will give as a result one of the eigenvalues \( a \) of the associated operator \( \hat{A} \), which satisfy the equation
\[
\hat{A}\psi = a\psi.
\]

3. **Expectation value.** For a system described by a normalized wavefunction \( \psi \), the expectation value of an observable \( A \) is given by
\[
\langle A \rangle = \int_{-\infty}^{\infty} \psi^* \hat{A} \psi \, dr.
\]

4. **Expansion in eigenfunctions.** The set of eigenfunctions of an operator \( \hat{A} \) forms a complete set of linearly independent functions. Therefore, an arbitrary state \( \psi \) can be expanded in the complete set of eigenfunctions of \( \hat{A} (\hat{A} \psi_n = a_n \psi_n) \), i.e.,
\[
\psi = \sum_n c_n \psi_n,
\]
where the sum may go to infinity. For the case where the eigenvalue spectrum is discrete and non-degenerate and where the system is in the normalized state \( \psi \), the probability of obtaining as a result of a measurement of \( A \) the eigenvalue \( a_n \) is \( |c_n|^2 \). This statement can be straightforwardly generalized to the degenerate and continuous spectrum cases.

Another **more general expression of this postulate** is ‘an arbitrary wavefunction can be expanded in a complete orthonormal set of eigenfunctions \( \psi_n \) of a set of commuting operators \( A_n \).’ It writes
\[
\psi = \sum_n c_n \psi_n,
\]
while the statement of orthonormality is
\[
\sum_n \int \psi^*_n(r,s,t) \psi(r,s,t) \, dr = \delta_{mn},
\]
where \( \delta_{mn} \) is the Kronecker symbol.
5. **Probability conservation.** The probability conservation is a consequence of the Hermitian property of $\hat{H}$. This property first implies that the norm of the state function is time independent and it also implies a local probability conservation which can be written (e.g., for a single particle without spin and with normalized wavefunction $\psi$) as

$$\frac{\partial}{\partial t}\rho(r, t) + \text{div} J(r, t) = 0,$$

where

$$J(r, t) = \frac{1}{m} \text{Re} \left[ \psi^* \left( \frac{i}{\hbar} \nabla \psi \right) \right].$$

6. **Reduction of the wave packet or projection hypothesis.** This statement does not need to be postulated since it can be deduced from other postulates. It is actually implicitly contained in von Neumann’s postulate.

### 7.3 Derived principles

1. **Heisenberg’s uncertainty principle.** If $P$ and $Q$ are two conjugate observables such that their commutator equals $i\hbar$, it is easy to show that their standard deviations $\Delta P$ and $\Delta Q$ satisfy the relation

$$\Delta P \Delta Q \geq \frac{\hbar}{2},$$

whatever the state function of the system. This applies to any couple of linear (but not necessarily Hermitian) operators and, in particular, to the couples of conjugate variables: position and momentum, time and energy. Moreover, generalized Heisenberg relations can be established for any couple of variables.

2. **The spin-statistic theorem.** When a system is composed of many identical particles, its physical states can only be described by state functions which are either completely antisymmetric (fermions) or completely symmetric (bosons) with respect to permutations of these particles, or, identically, by wavefunctions that change sign in a spatial reflection (fermions) or that remain unchanged in such a transformation (bosons). All half-spin particles are fermions and all integer-spin particles are bosons.

3. **The Pauli exclusion principle.** Two identical fermions cannot be in the same quantum state. This is a mere consequence of the spin-statistic theorem.
References


